Journal of Engineering Mathematics, Vol. 11, No. 4, October 1977
Noordhoff International Publishing–Leyden
Printed in The Netherlands

Two coplanar Griffith cracks in an infinite elastic layer under torsion

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(Received July 13, 1976)

SUMMARY

We consider the problem of determining the stress distribution in an infinitely long isotropic homogeneous elastic layer containing two coplanar Griffith cracks which are opened by internal shear stress acting along the lengths of the cracks. The faces of the layer are assumed to be stress free. The cracks are located in the middle plane of the layer parallel to its faces. By using Fourier transforms, we reduce the problem to the solution of a set of triple integral equations with a cosine kernel and a weight function. These equations are solved exactly by using finite Hilbert transform techniques. Finally we derive the closed form expressions for the stress intensity factors and the crack energy. Solutions to the following problems are derived as particular cases: (i) a single crack in an infinite layer under torsion, (ii) two coplanar cracks in an infinite space under torsion, (iii) a single crack in an infinite space under torsion.

1. Introduction

The problem of determining the stress distribution in an infinitely long elastic strip containing two collinear cracks lying in the middle line of the strip and opened by the internal pressure has been considered by Lowengrub and Srivastava [1] and Dhaliwal [2, 3]. These authors reduced the problem to the solution of a Fredholm integral equation of the second kind by using Fourier transforms, and then obtained its approximate solution by iteration assuming that the width of the strip is large in comparison to the length of the cracks.

In this paper we consider the problem of an elastic layer containing two coplanar cracks lying in the middle plane of the layer when the cracks are opened by shear loading along the length of the cracks and the layer surfaces are stress free. The solution of the problem is reduced to the solution of triple integral equations whose solution is obtained in the closed form. These triple integral equations are an extension of the type of triple integral equations studied by Singh [4, 5], and are more general than those discussed by Srivastava and Lowengrub [6]. A survey of the methods of solving triple integral equations may be found in Sneddon's book [7]. We obtained closed form expressions for the stress intensity factors and the crack energy for the general shear loading $p(x)$, as well as for the two special cases of $p(x) = S \cosh cx$ and $p(x) = S$. Solutions to the following problems are derived as particular cases: (i) a single crack in an infinite layer under torsion, (ii) two coplanar cracks in an infinite layer opened by shear loading, (iii) a single crack in an infinite space under torsion.

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2. Statement of the problem and derivation of the triple integral equations

Consider a rectangular cartesian coordinate system such that the cross-section of the layer is the strip $-\infty < x < \infty, -h < y < h$, and the cracks are located at $a < x < b, -b < x < -a, y = 0$. It is assumed that the edges of the strip $y = h, -h$ are stress free, and the cracks are opened by a variable shear stress acting on the faces of the cracks. Due to the symmetry about the x-axis, the problem can be solved by converting it into a mixed boundary value problem for the strip $-\infty < x < \infty, 0 < y < h$. In addition, if we assume that the shear loading on the cracks is an even function of x, the problem is further reduced to a semi-strip $0 < x < \infty$ and $0 < y < h$.

The non-zero displacement and stress components are given by

$$u_z = w(x, y), \quad \sigma_{xz} = \mu \frac{\partial w}{\partial x}, \quad \sigma_{yz} = \mu \frac{\partial w}{\partial y}, \tag{1}$$

where μ is the shear modulus of the material.

The equation of equilibrium reduces to

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) w = 0. \tag{2}$$

The solution of equation (2) may be taken in the form:

$$w(x, y) = \frac{1}{\mu} \left(\frac{2}{\pi} \right)^{\frac{1}{2}} F_c \left[\psi(\xi) \frac{\cosh(h - y)\xi}{\cosh h\xi}; \quad \xi \to x \right], \tag{3}$$

where, as usual, F_s and F_c denote the operators of the Fourier sine and cosine transform, respectively; hence

$$F_c[f(\xi, y); \xi \to x] = \left(\frac{2}{\pi} \right)^{\frac{1}{2}} \int_0^\infty f(\xi, y) \cos \xi x \, d\xi,$$

$$F_s[f(\xi, y); \xi \to x] = \left(\frac{2}{\pi} \right)^{\frac{1}{2}} \int_0^\infty f(\xi, y) \sin \xi x \, d\xi. \tag{4}$$

From (3) and (1) we find

$$w(x, 0) = \frac{1}{\mu} \left(\frac{2}{\pi} \right)^{\frac{1}{2}} F_c[\psi(\xi); \quad \xi \to x], \tag{5}$$

$$\sigma_{yz}(x, 0, z) = -\frac{d}{dx} \left(\frac{2}{\pi} \right)^{\frac{1}{2}} F_s[\psi(\xi) \tan h\xi; \quad \xi \to x]. \tag{6}$$

The boundary conditions for the problem are:

$$\begin{aligned} \sigma_{yz}(x, h, z) &= 0, & 0 < x < \infty, \\ \sigma_{yz}(x, 0, z) &= -p(x), & a < x < b, \\ u_z(x, 0, z) &= 0, & 0 < x < a, \ x > b, \end{aligned} \tag{7}$$

where $p(x)$ is an even function of x. The above boundary conditions are satisfied provided $\psi(\xi)$

satisfies the triple integral equations:

$$F_c[\psi(\xi);\quad \xi \to x] = 0, \qquad 0 < x < a,\ x > b, \tag{8}$$

$$\frac{d}{dx} F_s[\psi(\xi) \tanh h\xi;\quad \xi \to x] = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} p(x), \qquad a < x < b. \tag{9}$$

3. The solution of the triple integral equations

The solution of the triple integral equations of the above type has been discussed recently by Singh [8]. Following Singh, let us take

$$\psi(\xi) = \frac{1}{\xi} \int_a^b g(t^2) \cosh\left(\frac{\pi t}{2h}\right) \sin(\xi t)\, dt, \tag{10}$$

which satisfies (8), provided $g(t^2)$ satisfies the condition

$$\int_a^b g(t^2) \cosh\left(\frac{\pi t}{2h}\right) dt = 0. \tag{11}$$

Substituting for $\psi(\xi)$ from (10) into (9) and interchanging the order of integration, we obtain

$$\frac{d}{dx} \int_a^b g(t^2) \cosh(ct) \log\left|\frac{\sinh cx + \sinh ct}{\sinh cx - \sinh ct}\right| dt = \pi p(x), \qquad a < x < b, \tag{12}$$

where we have used the result (see Gradshteyn and Ryzhik [9], p. 516, 4.116(2))

$$\int_0^\infty \xi^{-1} \tanh(h\xi) \sin(\xi t) \sin(\xi x)\, dx = \frac{1}{2} \log\left|\frac{\sinh cx + \sinh ct}{\sinh cx - \sinh ct}\right|, \tag{13}$$

with $c = \pi/(2h)$. (14)

Interchanging the order of differentiation and integration in (12), we find that $g(t^2)$ must satisfy the integral equation

$$2c \int_a^b \frac{g(t^2) \sinh ct\, dt}{\cosh 2ct - \cosh 2cx} = \frac{\pi p(x)}{\cosh cx}, \qquad a < x < b. \tag{15}$$

Using the modified Tricomi theorem given by Singh [4], we find that the solution of the integral equation (15) is given by

$$\begin{aligned} g(t^2) = &-\frac{2}{h}\left[\frac{\cosh 2ct - \cosh 2ca}{\cosh 2cb - \cosh 2ct}\right]^{\frac{1}{2}} \int_a^b \left[\frac{\cosh 2cb - \cosh 2cx}{\cosh 2cx - \cosh 2ca}\right]^{\frac{1}{2}} \\ &\times \frac{p(x) \sinh cx}{\cosh 2cx - \cosh 2ct}\, dx \\ &+ \frac{2c^2 C_1}{[(\cosh 2ct - \cosh 2ca)(\cosh 2cb - \cosh 2ct)]^{\frac{1}{2}}}, \qquad a < t < b, \end{aligned} \tag{16}$$

where C_1 is an arbitrary constant.

Substituting for $g(t^2)$ from (16) into (11), we obtain:

$$C_1 = \frac{4}{\pi}\frac{\sinh cb}{F(\pi/2, q)}\int_a^b \left[\frac{\cosh 2cb - \cosh 2cx}{\cosh 2cx - \cosh 2ca}\right]^{\frac{1}{2}} \sinh cx\, p(x)dx$$
$$\times \int_a^b \left[\frac{\cosh 2ct - \cosh 2ca}{\cosh 2cb - \cosh 2ct}\right]^{\frac{1}{2}} \frac{\cosh ct}{\cosh 2cx - \cosh 2ct}\,dt, \tag{17}$$

where $F(\pi/2, q)$ denotes the elliptic integral of the first kind and

$$q = (\sinh cb)^{-1}(\sinh^2 cb - \sinh^2 ca)^{\frac{1}{2}}. \tag{18}$$

Since we have the identity

$$\left[\frac{(\cosh cx - \cosh ca)(\cosh cb - \cosh cy)}{(\cosh cb - \cosh cx)(\cosh cy - \cosh ca)}\right]^{\frac{1}{2}}\left[1 + \frac{\cosh cy - \cosh ct}{\cosh ct - \cosh ca}\right]$$
$$= \left[\frac{(\cosh cb - \cosh ct)(\cosh cy - \cosh ca)}{(\cosh ct - \cosh ca)(\cosh cb - \cosh cy)}\right]^{\frac{1}{2}}\left[1 - \frac{\cosh cy - \cosh ct}{\cosh cb - \cosh ct}\right], \tag{19}$$

we see that the solution (16), (17) may be written in the alternative form:

$$g(t^2) = -\frac{2}{h}\left[\frac{\cosh 2cb - \cosh 2ct}{\cosh 2ct - \cosh 2ca}\right]^{\frac{1}{2}}\int_a^b \left[\frac{\cosh 2cx - \cosh 2ca}{\cosh 2cb - \cosh 2cx}\right]^{\frac{1}{2}}$$
$$\times \frac{p(x)\sinh cx}{\cosh 2cx - \cosh 2ct}\,dx$$
$$+ \frac{2c^2 C_2}{[(\cosh 2ct - \cosh 2ca)(\cosh 2cb - \cosh 2ct)]^{\frac{1}{2}}}, \tag{20}$$

$$C_2 = \frac{4}{\pi}\frac{\sinh cb}{F(\pi/2, q)}\int_a^b \left[\frac{\cosh 2cb - \cosh 2ct}{\cosh 2ct - \cosh 2ca}\right]^{\frac{1}{2}} \cosh ct\, dt$$
$$\times \int_a^b \left[\frac{\cosh 2cx - \cosh 2ca}{\cosh 2cb - \cosh 2cx}\right]^{\frac{1}{2}} \frac{\sinh cx\, p(x)}{\cosh 2cx - \cosh 2ct}\,dx. \tag{21}$$

4. The stress intensity factors

Substituting for $\psi(\xi)$ from (10) into (6) and using (13), we obtain

$$[\sigma_{yz}(x, 0, z)]_{0<x<a} = -\frac{1}{h}\cosh cx \int_a^b \frac{g(t^2)\sinh 2ct}{\cosh 2ct - \cosh 2cx}\,dt, \tag{22}$$

$$[\sigma_{yz}(x, 0, z)]_{x>b} = \frac{1}{h}\cosh cx \int_a^b \frac{g(t^2)\sinh 2ct}{\cosh 2cx - \cosh 2ct}\,dt. \tag{23}$$

Substituting for $g(t^2)$ from equations (20) and (16) into the above equations, we obtain the expressions:

$$\{\sigma_{yz}(x, 0, z)]_{0<x<a} = \frac{2}{h}\cosh cx \left[\frac{\cosh 2cb - \cosh 2cx}{\cosh 2ca - \cosh 2cx}\right]^{\frac{1}{2}}$$

$$\times \int_a^b \left[\frac{\cosh 2cy - \cosh 2ca}{\cosh 2cb - \cosh 2cy}\right]^{\frac{1}{2}} \frac{p(y)\sinh cy}{\cosh 2cy - \cosh 2cx}\,dy$$

$$- 2c^2 C_2 \cosh cx[(\cosh 2ca - \cosh 2cx)(\cosh 2cb - \cosh 2cx)]^{-\frac{1}{2}}, \tag{24}$$

$$[\sigma_{yz}(x, 0, z)]_{x>b} = \frac{2}{h}\cosh cx \left[\frac{\cosh 2cx - \cosh 2ca}{\cosh 2cx - \cosh 2cb}\right]^{\frac{1}{2}}$$

$$\times \int_a^b \left[\frac{\cosh 2cb - \cosh 2cy}{\cosh 2cy - \cosh 2ca}\right]^{\frac{1}{2}} \frac{p(y)\sinh cy}{\cosh 2cx - \cosh 2cy}\,dy$$

$$+ 2c^2 C_1 \cosh cx[(\cosh 2cx - \cosh 2ca)(\cosh 2cx - \cosh 2cb)]^{-\frac{1}{2}}, \tag{25}$$

where, we have used the relation

$$\int_a^b \sin 2cu[(\cosh 2cu - \cosh 2ca)(\cosh 2cb - \cosh 2cu)]^{\frac{1}{2}}$$

$$\times (\cosh 2cu - \cosh 2cy)^{-1}\,du$$

$$= \begin{cases} h[(\cosh 2ca - \cosh 2cy)(\cosh 2cb - \cosh 2cy)]^{\frac{1}{2}}, & 0 < y < a, \\ 0, & a < y < b \\ -h[(\cosh 2cy - \cosh 2ca)(\cosh 2cy - \cosh 2cb)]^{\frac{1}{2}}, & y > b. \end{cases} \tag{26}$$

We now find the stress intensity factors in the form:

$$N_a = \lim_{x\to a-} [\{2(a - x)\}^{\frac{1}{2}}\sigma_{yz}(x, 0, z)]$$

$$= \frac{2}{\pi}\left(\frac{h}{\pi}\right)^{\frac{1}{2}} \frac{\cosh ca}{[\sinh 2ca(\sinh^2 cb - \sinh^2 ca)]^{\frac{1}{2}}} \left[2c(\sinh^2 cb - \sinh^2 ca)\right.$$

$$\left.\times \int_a^b \frac{p(y)\sinh cy}{[\sinh^2 cb - \sinh^2 cy)(\sinh^2 cy - \sinh^2 ca)]^{\frac{1}{2}}}\,dy - \pi c^2 C_2\right], \tag{27}$$

and

$$N_b = \lim_{x\to b+} [\{2(x - b)\}^{\frac{1}{2}}\sigma_{yz}(x, 0, z)]$$

$$= \frac{2}{\pi}\left(\frac{h}{\pi}\right)^{\frac{1}{2}} \frac{\cosh cb}{[\sinh 2cb(\sinh^2 cb - \sinh^2 ca)]^{\frac{1}{2}}} \left[2c(\sinh^2 cb - \sinh^2 ca)\right.$$

$$\left.\times \int_a^b \frac{p(y)\sinh cy}{[(\sinh^2 cy - \sinh^2 ca)(\sinh^2 cb - \sinh^2 cy)]^{\frac{1}{2}}}\,dy + \pi c^2 C_1\right], \tag{28}$$

where C_1 and C_2 are given by equations (17) and (21) respectively.

5. The crack energy

From equations (5) and (10), we find that the shape of the cracks is given by:

$$[u_z(x, 0, z)]_{a<x<b} = \frac{1}{\mu}\int_x^b g(t^2)\cosh ct\,dt. \tag{29}$$

The total energy required to open the crack is given by:

$$W = -2\int_a^b \sigma_{yz}(x, 0, z)u_z(x, 0, z)\,dx = 2\int_a^b p(x)u_z(x, 0, z)\,dx. \tag{30}$$

Hence from (29) and (30), we obtain:

$$W = \frac{2}{\mu}\int_a^b g(t^2)\cosh ct\,P(t)\,dt \tag{31}$$

where

$$P(t) = \int_a^t p(x)\,dx. \tag{32}$$

6. Solution for particular values of $p(x)$

Here we will obtain the closed form expressions for the stress intensity factors and the crack energy for the following particular values of $p(x)$:

Case (a):

$$p(x) = S\cosh cx, \tag{33}$$

where S is a constant.

Substituting for $p(x)$ from (33) into (16) and (20) and using the following well-known result

$$\int_a^b \frac{c(\cosh ct - \cosh ca)^v(\cosh cb - \cosh ct)^{-v}}{\cosh ct - \cosh cy}\sinh ct\,dt$$
$$= \pi\operatorname{cosec} v\pi\left[1 - \cos v\pi\left(\frac{\cosh cy - \cosh ca}{\cosh cb - \cosh cy}\right)^v\right], \quad |v| < 1,\ a < y < b, \tag{34}$$

we obtain

$$g(t^2) = -S\left[\frac{\cosh 2cb - \cosh 2ct}{\cosh 2ct - \cosh 2ca}\right]^{\frac{1}{2}}$$
$$+ 2c^2C_2[(\cosh 2ct - \cosh 2ca)(\cosh 2cb - \cosh 2ct)]^{-\frac{1}{2}}, \tag{35}$$

or

$$g(t^2) = S\left[\frac{\cosh 2ct - \cosh 2ca}{\cosh 2cb - \cosh 2ct}\right]^{\frac{1}{2}}$$
$$+ 2c^2C_1[(\cosh 2ct - \cosh 2ca)(\cosh 2cb - \cosh 2ct)]^{-\frac{1}{2}}. \tag{36}$$

From (33), (17) and (21), we obtain

$$\left.\begin{aligned} C_1 &= [S/(c^2F)](F\sinh^2 ca - E\sinh^2 cb), \\ C_2 &= (S/c^2)(1 - E/F)\sinh^2 cb, \end{aligned}\right\} \tag{37}$$

where $F = F(\pi/2, q)$ and $E = E(\pi/2, q)$ are respectively the elliptic integrals of the first and second kind and q is given by equation (18).

Hence we may write

$$\begin{aligned} g(t^2) = {} & S[(\sinh^2 ct - \sinh^2 ca)(\sinh^2 cb - \sinh^2 ct)]^{-\frac{1}{2}} \\ & \times [\sinh^2 ct - (E/F)\sinh^2 cb]. \end{aligned} \tag{38}$$

From (22), (23), (35), (36) and (37), we obtain

$$\begin{aligned} [\sigma_{yz}(x, 0, z)]_{0<x<a} = {} & S\cosh cx[(E/F)\sinh^2 cb - \sinh^2 cx] \\ & \times [(\sinh^2 ca - \sinh^2 cx)(\sinh^2 cb - \sinh^2 cx)]^{-\frac{1}{2}}, \end{aligned} \tag{39}$$

$$\begin{aligned} [\sigma_{yz}(x, 0, z)]_{x>b} = {} & S\cosh cx[\sinh^2 cx - (E/F)\sinh^2 cb] \\ & \times [(\sinh^2 cx - \sinh^2 ca)(\sinh^2 cx - \sinh^2 cb)]^{-\frac{1}{2}}. \end{aligned} \tag{40}$$

We obtain the following expressions for the stress intensity factors N_a and N_b respectively from equations (27), $(37)_2$ and equations (28), $(37)_1$:

$$\begin{aligned} N_a = {} & \left(\frac{2}{c}\right)^{\frac{1}{2}} S\cosh(ca)[\sinh 2ca(\sinh^2 cb - \sinh^2 ca)]^{-\frac{1}{2}} \\ & \times [(E/F)\sinh^2 cb - \sinh^2 ca], \end{aligned} \tag{41}$$

$$N_b = \left(\frac{2}{c}\right)^{\frac{1}{2}} S(1 - E/F)\cosh cb \sinh^2 cb[\sinh 2cb(\sinh^2 cb - \sinh^2 ca)]^{-\frac{1}{2}}, \tag{42}$$

where we have used the integral

$$c\int_a^b \sinh(2cy)[(\sinh^2 cb - \sinh^2 cy)(\sinh^2 cy - \sinh^2 ca)]^{-\frac{1}{2}} dy = \pi. \tag{43}$$

Substituting for $g(t^2)$ from (38) into (29) and making use of the following integrals

$$c\int_x^b \cosh(ct)[(\sinh^2 ct - \sinh^2 ca)(\sinh^2 cb - \sinh^2 ct)]^{-\frac{1}{2}} dt = F_2(\lambda, q)/\sinh(cb),$$

$$\begin{aligned} & c\int_x^b \cosh(ct)\left[\frac{\sinh^2 ct - \sinh^2 ca}{\sinh^2 cb - \sinh^2 ct}\right]^{\frac{1}{2}} dt \\ & = \sinh(cb)\left[E_2(\lambda, q) - \frac{\sinh^2 ca}{\sinh^2 cb} F_2(\lambda, q)\right], \end{aligned} \tag{44}$$

which can be obtained from Gradshteyn and Ryzhik [9], we obtain

$$[u_z(x, 0, z)]_{a<x<b} = \frac{S}{\mu c}[\sinh(cb)(E_2 - F_2E/F)], \tag{45}$$

where F_2 and E_2 are elliptic integrals of the first and second kind respectively and

$$\lambda = \sin^{-1}\left\{\left[\frac{\sinh^2 cb - \sinh^2 cx}{\sinh^2 cb - \sinh^2 ca}\right]^{\frac{1}{2}}\right\}. \tag{46}$$

From (31), (32) and (33), we find that the total work done to open the crack is given by

$$W = \frac{2S}{\mu c}\int_a^b g(t^2)\cosh(ct)(\sinh ct - \sinh ca)\,dt. \tag{47}$$

Substituting for $g(t^2)$ from (38) into (47), we find that

$$W = (I_1 - I_2)S^2/(\mu c), \tag{48}$$

where

$$I_1 = \int_a^b \sinh(2ct)[\sinh^2 ct - (E/F)\sinh^2 cb]$$
$$\times [(\sinh^2 ct - \sinh^2 ca)(\sinh^2 cb - \sinh^2 ct)]^{-\frac{1}{2}}dt$$

$$I_2 = 2\int_a^b \cosh(ct)\sinh(ca)[\sinh^2 ct - (E/F)\sinh^2 cb]$$
$$\times [(\sinh^2 ct - \sinh^2 ca)(\sinh^2 cb - \sinh^2 ct)]^{-\frac{1}{2}}dt.$$

Evaluating I_1 and I_2 by making use of the integrals (44) for $x = a$, we find that

$$I_2 \equiv 0$$

and hence

$$W = \frac{I_1 S^2}{\mu c} = \frac{\pi S^2}{2\mu c^2}[\sinh^2 ca + (1 - 2E/F)\sinh^2 cb]. \tag{49}$$

Case (b):

$$p(x) = S$$

In this case, we find from (27) and (28) that

$$N_a = \frac{1}{\pi}\left(\frac{2}{c}\right)^{\frac{1}{2}}\cosh(ca)[\sinh(2ca)(\sinh^2 cb - \sinh^2 ca)]^{-\frac{1}{2}}$$
$$\times\left[2SF\left(\frac{\pi}{2}, q_1\right)(\cosh cb)^{-1}(\sinh^2 cb - \sinh^2 ca) - 2c^2C_2\right], \tag{51}$$

$$N_b = \frac{1}{\pi}\left(\frac{2}{c}\right)^{\frac{1}{2}}\cosh(cb)[\sinh(2cb)(\sinh^2 cb - \sinh^2 ca)]^{-\frac{1}{2}}$$
$$\times\left[2SF\left(\frac{\pi}{2}, q_1\right)(\cosh cb)^{-1}(\sinh^2 cb - \sinh^2 ca) + 2c^2C_1\right], \tag{52}$$

where

$$q_1 = (\cosh cb)^{-1}(\cosh^2 cb - \cosh^2 ca)^{\frac{1}{2}} \tag{53}$$

and C_1 and C_2 may be determined from (17), (21) and (50).

7. Derivation of the solution of particular problems

In this section we will derive solution of the following particular problems:

(*i*) *A single crack in an infinite layer under torsion:*

If we have a single crack located at $-b < x < b$, $y = 0$, in the layer $-\infty < x < \infty$, $-h < y < h$, we let $a \to 0$ in (28), (31) and (32) to obtain the following expressions for the stress intensity factor and crack energy for this problem:

$$N_b = \frac{1}{\pi}(2c \sinh 2cb)^{\frac{1}{2}}\left[\int_0^b p(y)(\sinh^2 cb - \sinh^2 cy)^{-\frac{1}{2}} dy \right.$$
$$\left. + \pi c \coth cb(\sinh 2cb)^{-1} C_1\right], \tag{54}$$

$$W = \frac{2}{\mu}\int_0^b g(t^2)\cosh ct\, P(t) dt, \quad P(t) = \int_0^t p(x) dx, \tag{55}$$

where $g(t^2)$ and C_1 are given by (16), (17) and (18) by taking $a = 0$.

For $p(x) = S \cosh cx$, we find

$$N_b = S(\sinh(2cb)/2c)^{\frac{1}{2}}, \tag{56}$$

$$W = \frac{\pi}{2\mu c^2}\sinh^2 cb. \tag{57}$$

And for $p(x) = S$, we find

$$N_b = \frac{1}{\pi}\left(\frac{c}{2}\sinh 2cb\right)^{-\frac{1}{2}}\left[2SF\left(\frac{\pi}{2}, \tanh cb\right)\sinh cb + 2c^2 C_1 \coth cb\right],$$

where C_1 may be determined from (17) for $a = 0$ and $p(x) = S$.

(*ii*) *Two coplanar cracks in an infinite plane under torsion:*

If the cracks are located at $a < |x| < b$, $y = 0$ in an infinite plane $-\infty < x < \infty$, $-\infty < y < \infty$, we let $h \to \infty$ (i.e. $c \to 0$) in the results of Sections 4 and 5 to obtain

$$N_a = \frac{2}{\pi}\left(\frac{b^2 - a^2}{a}\right)^{\frac{1}{2}}\int_a^b \frac{y\, p(y) dy}{\sqrt{(b^2 - y^2)(y^2 - a^2)}} - \frac{C_2}{\sqrt{a(b^2 - a^2)}}, \tag{59}$$

$$N_b = \frac{2}{\pi}\left(\frac{b^2 - a^2}{b}\right)^{\frac{1}{2}}\int_a^b \frac{y\, p(y) dy}{\sqrt{(b^2 - y^2)(y^2 - a^2)}} + \frac{C_1}{\sqrt{b(b^2 - a^2)}}, \tag{60}$$

$$W = \frac{2}{\mu}\int_a^b g(t^2) P(t) dt, \quad P(t) = \int_a^t p(x) dx, \tag{61}$$

where

$$g(t^2) = \frac{-2}{\pi}\left(\frac{t^2-a^2}{b^2-t^2}\right)^{\frac{1}{2}} \int_a^b \left(\frac{b^2-x^2}{x^2-t^2}\right)^{\frac{1}{2}} \frac{xp(x)}{x^2-t^2}dx + \frac{C_1}{\sqrt{(t^2-a^2)(b^2-t^2)}},$$

or

$$g(t^2) = -\frac{2}{\pi}\left(\frac{b^2-t^2}{t^2-a^2}\right)^{\frac{1}{2}} \int_a^b \left(\frac{x^2-a^2}{b^2-x^2}\right)^{\frac{1}{2}} \frac{xp(x)}{x^2-t^2}dx + \frac{C_2}{\sqrt{(t^2-a^2)(b^2-t^2)}},$$

$$C_1 = \frac{2}{\pi}\frac{b}{F(\pi/2, \sqrt{1-b^2/a^2})}\int_a^b \left(\frac{b^2-x^2}{x^2-a^2}\right)^{\frac{1}{2}} xp(x)dx \int_a^b \left(\frac{t^2-a^2}{b^2-t^2}\right)\frac{dt}{x^2-t^2},$$

$$C_2 = \frac{2}{\pi}\frac{b}{F(\pi/2, \sqrt{1-b^2/a^2})}\int_a^b \left(\frac{b^2-t^2}{t^2-a^2}\right)^{\frac{1}{2}} dt \int_a^b xp(x)\left(\frac{x^2-a^2}{b^2-x^2}\right)^{\frac{1}{2}}\frac{dx}{x^2-t^2}.$$

For $p(x) = S$, we obtain

$$N_a = S[b^2E(\pi/2, \sqrt{1-a^2/b^2})/F(\pi/2, \sqrt{1-a^2/b^2}) - a^2]/\sqrt{a(b^2-a^2)}, \tag{62}$$

$$N_b = Sb^2[1 - E(\pi/2, \sqrt{1-a^2/b^2})/F(\pi/2, \sqrt{1-a^2/b^2})]/\sqrt{b(b^2-a^2)}, \tag{63}$$

$$W = \frac{\pi S^2}{2\mu}[a^2 + b^2 - 2b^2E(\pi/2, \sqrt{1-a^2/b^2})/F(\pi/2, \sqrt{1-a^2/b^2})]. \tag{64}$$

(*iii*) *A single crack in an infinite plane under torsion:*

If a single crack is located at $-b < x < b$, $y = 0$, in an infinite plane $-\infty < x < \infty$, $-\infty < y < \infty$, we let $h \to \infty$ (i.e. $c \to 0$) in the results obtained in (i) above or put $a = 0$ in the results (60) and (61) and obtain:

$$N_b = \frac{2}{\pi}\sqrt{b}\int_0^b \frac{p(y)}{\sqrt{b^2-y^2}}dy, \qquad (C_1 \to 0), \tag{65}$$

$$W = -\frac{4}{\pi\mu}\int_0^b \frac{t\,P(t)}{\sqrt{b^2-t^2}}\int_0^b \frac{\sqrt{b^2-x^2}}{x^2-t^2}p(x)dxdt, \quad P(t) = \int_0^t p(x)dx, \tag{66}$$

for a single crack of length $2b$. When the shear loading is constant (i.e. $p(x) = S$), we obtain

$$N_b = S\sqrt{b}, \quad W = \frac{\pi S^2 b^2}{2\mu} = \frac{(1+\eta)}{2E}S^2b^2, \tag{67}$$

where E is the Young's modulus and η is the Poisson's ratio of the elastic material. The expression (67) for the crack energy agrees with the one given by Sneddon and Lowengrub [10, p. 38] except for a factor 2 in the denominator which obviously is a printing mistake in their result.

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